BPS Equations in Ω-deformed $\mathcal{N} = 4$ Super Yang-Mills Theory

Hiroaki Nakajima (Sichuan University)

in collaboration with

Katsushi Ito, **Yusuke Kanayama** (Tokyo Tech)

and **Shin Sasaki** (Kitasato Univ.)

Nov. 16, 2015 @ NCTS, Hsinchu

contents

1. Introduction

2. $\mathcal{N} = 4$ super Yang-Mills in Ω-background with NS limit

3. Central charges and BPS bounds

4. BPS equations

5. Summary

based on arXiv:1509.07381 to appear in JHEP

1. Introduction

- Ω-background

- 4D spacetime fiber on internal base space is twisted by the action of spacetime rotation.

- used to regularize the IR divergence in instanton calculus (giving mass terms to the instanton moduli)

- breaks all of translation symmetry and preserves a part of rotation symmetry

- also preserves a part of supersymmetry

- Ω-background with Nekrasov-Shatashvili(NS) limit

- Only the fiber of 2D subspace is twisted.

- 2D Poincaré symmetry is recovered.

- Unbroken SUSYs are enhanced.

- well-defined BPS spectra

- effective 2D theory $\leftrightarrow$ integrable systems

  [Nekrasov-Witten], [Bulycheva-Chen-Gorsky-Koroteev], etc.

- quantization of SW curve

Here we consider the Ω-deformation of $\mathcal{N} = 4$ super Yang-Mills theory

- Maximally supersymmetric gauge theory
- More BPS objects, e.g. quarter-BPS dyons
  [Bergman], [Lee-Yi], [Hata-Hashimoto-Sasakura], etc.
- Montonen-Olive electric-magnetic duality

2. $\mathcal{N}=4$ super Yang-Mills in Ω-background with NS limit

$$
\begin{array}{c}
\text{4D } \mathcal{N}=4 \text{ super Yang-Mills} \\
\Big\uparrow \text{dimensional reduction} \\
\text{10D } \mathcal{N}=(1,0) \text{ super Yang-Mills}
\end{array}
$$

- 10D Ω-background metric with NS limit
  $(\mathbb{R}^{3,1} \times \mathrm{T}^6 \to \mathbb{R}^{1,1} \times (\mathbb{R}^2 \times \mathrm{T}^6)_\epsilon\ )$

$$
ds^2 = -(dx^0)^2 + (dx^3)^2 + (dx^1 + \epsilon_a x^2 dy^a)^2 + (dx^2 - \epsilon_a x^1 dy^a)^2 + dy^a dy^a, \quad a = 1, \ldots, 6.
$$

R-symmetry Wilson line: introduced as the contorsion in 10D

$$\mathcal{A}_{bc} = \mathcal{A}_{a,bc}\, dx^a, \quad \mathcal{A}_{a,bc}: \text{ constant}$$

10D Lagrangian ($\partial_{\mathcal{M}} \to \widehat{\nabla}_{\mathcal{M}} = \partial_{\mathcal{M}} + $ (torsionful connection))

$$\mathcal{L}_{\rm 10D} = \frac{1}{\kappa g_{10}^2} \mathrm{Tr}\left[-\frac{1}{4} e (e^{\mathcal{M}}{}_M e^{\mathcal{N}}{}_N \widehat{F}_{\mathcal{M}\mathcal{N}})^2 - \frac{i}{2} e \bar{\Psi} \Gamma^M e^{\mathcal{M}}{}_M \nabla^{(G)}_{\mathcal{M}} \Psi \right].$$

Here

$$\widehat{F}_{\mathcal{M}\mathcal{N}} = F_{\mathcal{M}\mathcal{N}} - T_{\mathcal{M}\mathcal{N}}{}^P e^{\mathcal{P}}{}_P A_{\mathcal{P}},$$
$$\nabla^{(G)}_{\mathcal{M}} = \widehat{\nabla}_{\mathcal{M}} + i[A_{\mathcal{M}}, *],$$

4D Lagrangian

$$
\begin{aligned}
\mathcal{L} = \frac{1}{\kappa g^2}\,\mathrm{Tr}\Big[& -\frac{1}{4}F_{mn}F^{mn} + i\bar{\Lambda}_A\bar{\sigma}^m D_m\Lambda^A - \frac{1}{2}(D_m\varphi_a - F_{mn}\Omega_a^n)^2 \\
& + \frac{1}{2}(\bar{\Sigma}^a)_{AB}\Lambda^A[\varphi_a,\Lambda^B] + \frac{1}{2}(\Sigma^a)^{AB}\bar{\Lambda}_A[\varphi_a,\bar{\Lambda}_B] \\
& + \frac{i}{2}\Omega_a^m\big((\bar{\Sigma}^a)_{AB}\Lambda^A D_m\Lambda^B + (\Sigma^a)^{AB}\bar{\Lambda}_A D_m\bar{\Lambda}_B\big) \\
& - \frac{i}{4}\Omega_{mna}\big((\bar{\Sigma}^a)_{AB}\Lambda^A\sigma^{mn}\Lambda^B + (\Sigma^a)^{AB}\bar{\Lambda}_A\bar{\sigma}^{mn}\bar{\Lambda}_B\big) \\
& + \frac{i}{8}T_{ab,c}\big((\bar{\Sigma}^{abc})_{AB}\Lambda^A\Lambda^B + (\Sigma^{abc})^{AB}\bar{\Lambda}_A\bar{\Lambda}_B\big) \\
& - \frac{1}{4}\Big(i[\varphi_a,\varphi_b] - \Omega_a^m D_m\varphi_b + \Omega_b^m D_m\varphi_a + \Omega_a^m\Omega_b^n F_{mn} - T_{ab}{}^c\varphi_c\Big)^2\Big],
\end{aligned}
$$

$$
T_{ab}{}^c = -\mathcal{A}_{a,b}{}^c + \mathcal{A}_{b,a}{}^c, \quad \mathcal{A}_{a,bc} = -\frac{1}{2}(T_{ab,c} - T_{bc,a} + T_{ca,b}).
$$

Here

$$\Omega_a^m = \Omega^m{}_{na} x^m, \quad \Omega^m{}_{na} = \begin{pmatrix} 0 & \epsilon_a & 0 & 0 \\ -\epsilon_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad m, n = 1, 2, 3, 0,$$

$$\Gamma^M = \left(\begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \otimes \mathbf{1}_8, \ \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Sigma^a \\ \bar{\Sigma}^a & 0 \end{pmatrix} \right).$$

remaining parameters after imposing SUSY condition

4 SUSY case: ϵ_1, ϵ_2, $m_1 = \mathcal{A}_{1,34} + \mathcal{A}_{1,56}$, $m_2 = \mathcal{A}_{2,34} + \mathcal{A}_{2,56}$.

Ω-deformation of $\mathcal{N} = 2^*$ theory

m_1, m_2: mass parameters.

8 SUSY case: ϵ_1, ϵ_2, ϵ_5, ϵ_6,

Four parameters can be nonvanishing. these two cases have overlap as

$$m_{1,2} = \epsilon_{1,2}/2 \text{ in 4 SUSY case} \quad = \quad \epsilon_5, \epsilon_6 = 0 \text{ in 8 SUSY case}$$

3. Central charges and BPS bounds

• SUSY algebra

supercurrent

$$j_\zeta^{\mathcal{M}} = \frac{1}{\kappa g_{10}^2} \mathrm{Tr}\left[\frac{i}{2} e \widehat{F}_{\mathcal{N}\mathcal{P}} \bar{\Psi} \Gamma^{\mathcal{M}} \Gamma^{\mathcal{N}\mathcal{P}} \zeta \right].$$

SUSY variation of supercurrent

$$\delta_\xi j_\zeta^{\mathcal{M}} = -i[\bar{\xi}\mathcal{Q}, j_\zeta^{\mathcal{M}}], \quad \bar{\xi}\mathcal{Q} = \int d^9 x j_\xi^0.$$

spatial integration of temporal component → SUSY algebra

SUSY algebra in 10D formulation

$$[\bar{\xi}\mathcal{Q},\bar{\zeta}\mathcal{Q}] = 2\bar{\xi}\Gamma^{M}\zeta\ P_M + \frac{1}{4}\bar{\xi}\Gamma^{MNPQR}\zeta\ \mathcal{Z}_{MNPQR}\ ,$$

$$\mathcal{Z}_{MNPQR} = \int d^9x\,\frac{1}{\kappa g_{10}^2}\mathrm{Tr}\left[\frac{1}{5!}\varepsilon_{MNPQR}{}^{STUVW}ee^0{}_S\widehat{F}_{TU}\widehat{F}_{VW}\right],$$

Reduction to 4D

$$[\xi^{\alpha A}Q_{\alpha A}+\bar{Q}_{\dot{\alpha}}{}^{A}\bar{\xi}^{\dot{\alpha}}{}_{A},\ \zeta^{\beta B}Q_{\beta B}+\bar{Q}_{\dot{\beta}}{}^{B}\bar{\zeta}^{\dot{\beta}}{}_{B}]$$

$$= 2(\xi\sigma^m\bar{\zeta}+\bar{\xi}\bar{\sigma}^m\zeta)P_m + 2(\xi\bar{\Sigma}_a\zeta-\bar{\xi}\Sigma_a\bar{\zeta})P_a - 2i(\xi\bar{\Sigma}_a\zeta+\bar{\xi}\Sigma_a\bar{\zeta})\left(\frac{1}{5!}\varepsilon^{abcdef}Z_{bcdef}\right)$$

$$- i(\xi\sigma^i\bar{\Sigma}_{abcd}\bar{\zeta}+\bar{\xi}\bar{\sigma}^i\Sigma_{abcd}\zeta)Z_{iabcd} + 2(\xi\sigma^{ij}\bar{\Sigma}_{abc}\zeta+\bar{\xi}\bar{\sigma}^{ij}\Sigma_{abc}\bar{\zeta})Z_{ijabc}$$

$$+ \frac{i}{4}\varepsilon^{ijk}(\xi\sigma^0\bar{\Sigma}_{ab}\bar{\zeta}-\bar{\xi}\bar{\sigma}^0\Sigma_{ab}\zeta)Z_{ijkab},$$

- $P_a = q_a^{(e)}$ (electric charge)

  Using Gauss' law, $q_a^{(e)}$ can be expressed as

$$q_a^{(e)} = \int d^3x\, \partial_i \mathrm{Tr}\big[\varphi_a(E_i - \Omega_{ib} D_0 \varphi_b + \Omega_{ib}\Omega_{jb} E_j)\big]$$

$$+ \text{(angular momentum)} + \text{(R-charge)}$$

- $*Z_{bcdef} = q_a^{(m)} = \int d^3x\, \partial_i \mathrm{Tr}\big[\varphi_a B_i\big]$ (magnetic charge)

  No Ω-deformation

- $(Z_{iabcd}, Z_{ijabc}, Z_{ijkab})$

  Central charge for (vortices, walls, space-filling objects). However they all vanish under vacuum boundary condition.

• SUSY algebra and BPS bounds for 8 SUSY case

$\Rightarrow$ same as $\mathcal{N} = (4, 4)$ 2D SUSY algebra

$$\begin{aligned}
\{Q_{11}, \bar{Q}_{\dot{1}}{}^{1}\} &= \{Q_{13}, \bar{Q}_{\dot{1}}{}^{3}\} & \{Q_{22}, \bar{Q}_{\dot{2}}{}^{2}\} &= \{Q_{24}, \bar{Q}_{\dot{2}}{}^{4}\} \\
&= 2(P^0 + P^3), & &= 2(P^0 - P^3), \\
\{Q_{11}, Q_{22}\} &= 2i(q_1 + iq_2), & \{\bar{Q}_{\dot{1}}{}^{1}, \bar{Q}_{\dot{2}}{}^{2}\} &= -2i(\bar{q}_1 - i\bar{q}_2), \\
\{Q_{13}, Q_{22}\} &= 2i(q_5 + iq_6), & \{\bar{Q}_{\dot{1}}{}^{3}, \bar{Q}_{\dot{2}}{}^{2}\} &= -2i(\bar{q}_5 - i\bar{q}_6), \\
\{Q_{13}, Q_{24}\} &= 2i(q_1 - iq_2), & \{\bar{Q}_{\dot{1}}{}^{3}, \bar{Q}_{\dot{2}}{}^{4}\} &= -2i(\bar{q}_1 + i\bar{q}_2), \\
\{Q_{11}, Q_{24}\} &= -2i(q_5 - iq_6), & \{\bar{Q}_{\dot{1}}{}^{1}, \bar{Q}_{\dot{2}}{}^{4}\} &= 2i(\bar{q}_5 + i\bar{q}_6),
\end{aligned}$$

complexified charges

$$q_a = q_a^{(e)} - iq_a^{(m)}, \quad a = 1, 2, 5, 6.$$

BPS mass formula

$$M = \sqrt{|q^{(e)}|^2 + |q^{(m)}|^2 \pm 2|q^{(e)}||q^{(m)}| \sin\alpha},$$

$|q^{(e)}|$, $|q^{(m)}|$: magnitudes of vectors $q^{(e)} = (q_1^{(e)},\, q_2^{(e)},\, q_5^{(e)},\, q_6^{(e)})$
and $q^{(m)} = (q_1^{(m)},\, q_2^{(m)},\, q_5^{(m)},\, q_6^{(m)})$

α: angle between $q^{(e)}$ and $q^{(m)}$

$\sin\alpha = 0$: half-BPS (4 SUSY)

$\sin\alpha \neq 0$: quarter-BPS (2 SUSY)

4. BPS equations for 8 SUSY case

- half-BPS

preserved SUSY: $Q_{11} - e^{i\theta}\bar{Q}_{\dot{2}}{}^{2}$, $Q_{13} - e^{i\theta}\bar{Q}_{\dot{2}}{}^{4}$ and their conjugates

BPS dyon equations

$$
\begin{aligned}
&E_i + (D_i\varphi_1 + \Omega_1^m F_{mi})\sin\theta = 0,\\
&B_i + (D_i\varphi_1 + \Omega_1^m F_{mi})\cos\theta - i\delta_{3,i}[\varphi_3, \varphi_4] = 0,\\
&D_0\varphi_1 + \Omega_1^m F_{m0} = 0,\\
&D_m\varphi_a + \Omega_a^n F_{nm} = 0, \quad (a = 2, 5, 6),\\
&(D_1 + iD_2)(\varphi_3 - i\varphi_4) = 0,
\end{aligned}
$$

and

$$D_3\varphi_b + i(-1)^{f(b)}\Big([\varphi_1, \varphi_{f(b)}] + i\Omega_1^m D_m\varphi_{f(b)} - i(-1)^{f(b)}\epsilon_1^1\varphi_b\Big)\cos\theta = 0,$$

$$D_0\varphi_b + i\Big([\varphi_1, \varphi_b] + i\Omega_1^m D_m\varphi_b + i(-1)^{f(b)}\epsilon_1^1\varphi_{f(b)}\Big)\sin\theta = 0, \ (b = 3, 4),$$

$$H_{cd} = 0, \quad \big(1 \leq c < d \leq 6, \ (c, d) \neq (1, 3), (1, 4), (3, 4)\big),$$

where

$$f(3) = 4, \quad f(4) = 3, \quad f(5) = 6, \quad f(6) = 5,$$

$$g(3) = 0, \quad g(4) = 0, \quad g(5) = 1, \quad g(6) = 1.$$

$$H_{ab} = i[\varphi_a, \varphi_b] - \Omega_a^m D_m\varphi_b + \Omega_b^m D_m\varphi_a + \Omega_a^m\Omega_b^n F_{mn} - T_{ab}{}^c\varphi_c.$$

• quarter-BPS

preserved SUSY: $Q_{11} - e^{i\theta}\bar{Q}_{\dot{2}}{}^{2}$ and its conjugate

BPS dyon equations

$$E_i - (D_i\phi_2 + \omega_2^m F_{mi}) = 0,$$

$$B_i + (D_i\phi_1 + \omega_1^m F_{mi})$$

$$- i\delta_{3,i}\left([\varphi_3, \varphi_4] - [\varphi_5, \varphi_6] - i\Omega_5^m D_m\varphi_6 + i\Omega_6^m D_m\varphi_5\right) = 0,$$

$$D_0\phi_1 + \omega_1^m F_{m0} - i[\phi_1, \phi_2] + \omega_1^m D_m\phi_2 - \omega_2^m D_m\phi_1 = 0,$$

$$D_0\phi_2 + \omega_2^m F_{m0} = 0,$$

$$(D_1 + iD_2)(\varphi_3 - i\varphi_4) = 0,$$

$$(D_1 + iD_2)(\varphi_5 + i\varphi_6) + (\Omega_5^m + i\Omega_6^m)(F_{m1} + iF_{m2}) = 0,$$

and

$$D_3\varphi_a + i(-1)^{f(a)}\Big([\phi_1, \varphi_{f(a)}] + i\omega_1^m D_m\varphi_{f(a)} - i(-1)^{f(a)}\varepsilon_1^1\varphi_a\Big) = 0,$$

$$D_0\varphi_a + i[\phi_2, \varphi_a] - \omega_2^m D_m\varphi_a + (-1)^{f(a)}\varepsilon_1^1\varphi_{f(a)} = 0, \quad (a = 3, 4),$$

$$D_3\varphi_b + \Omega_b^m F_{m3} + i(-1)^{f(b)}\Big([\phi_1, \varphi_{f(b)}] + i\omega_1^m D_m\varphi_{f(b)} - i\Omega_{f(b)}^m D_m\phi_2\Big)$$
$$= 0,$$

$$D_0\varphi_b + \Omega_b^m F_{m0} + i[\phi_2, \varphi_b] - \omega_2^m D_m\varphi_b + \Omega_b^m D_m\phi_2 = 0, \quad (b = 5, 6),$$

$$[\varphi_3, \varphi_6] - [\varphi_4, \varphi_5] - i\Omega_6^m D_m\varphi_3 + i\Omega_5^m D_m\varphi_4 - i\epsilon_5^1\varphi_3 - i\epsilon_6^1\varphi_4 = 0,$$

$$[\varphi_3, \varphi_5] + [\varphi_4, \varphi_6] - i\Omega_5^m D_m\varphi_3 - i\Omega_6^m D_m\varphi_4 + i\epsilon_6^1\varphi_3 - i\epsilon_5^1\varphi_4 = 0,$$

5. Summary

summary

1. In Ω-deformed $\mathcal{N}=4$ super Yang-Mills with NS limit, we only have the electric and the magnetic charges as the central charge if we adopt with the vacuum boundary conditon. Other charges (for vortices etc.) are vanishing.

2. This imples that we have only dyons as the BPS solitons with the finite energy.

3. We have obtained the equation of the BPS dyons which preserve the half or the quarter of the supersymmetry.

outlook

- Solutions

  $\mathcal{N}=2$ single monopole case: [Ito-Kamoshita-Sasaki]

  Reduced to the Ernst equation (integrable)

- Nahm construction and moduli space

- Solution with finite energy density?

- Dyonic instantons in 5D

- $\mathcal{N}=2$ with fundamental hypers $\rightarrow$ vortices

  [Bulycheva-Chen-Gorsky-Koroteev], [Tong-Turner]