Exploring spectral bounds via toroidal compactifications

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CFTs are great

Plenty of reasons to study CFTs:

Direct physical applications,
signposts in space of QFTs,
AdS/CFT applications, ….

LOTS of recent success via conformal bootstrapping:

constructing CFTs using (1) conf. inv, (2) unitarity, (3) OPE associativity

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

$$v^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(u,v) = u^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(v,u)$$

$$0 = \underbrace{F_{0,0}(u,v)}_{\text{unit op.}} + \underbrace{\sum \lambda_{\mathcal{O}}^2 F_{\Delta,\ell}(u,v)}_{\text{everything else}}$$

Details from D. Poland...

Modular bootstrapping

Interested in 2d CFTs with c>1 (string theory, phase transitions, AdS/CFT..)

Lose a lot of power of local symmetry when c>1; bootstrapping mostly the same

Use another principle in 2d to help better constrain theories:

modular invariance of partition function on torus

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - \bar{c}/24}\right) \qquad q = \exp(2\pi i \tau) \qquad \tau \equiv (K^1 + i\beta)/2\pi$$

(CFT defined on all Riemann surfaces iff 4-pt crossing symm.n sphere AND modular invariance of *Z* and 1-pt fcns on torus) [Moore, Seiberg '88]

Modular group = 2x2 unimodular matrix of integers $PSL(2,\mathbb{Z})$:

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad ad - bc = 1$$

$$T : \tau \to \tau + 1 \quad or \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S : \tau \to -\frac{1}{\tau} \quad or \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Modular invariance constraints

Previously:

Bounds on primary conformal dims Δ_n ($c_{L,R} > 1$) [0902.2790,1307.6562, 1511.04074]

Bounds on number of primaries,states [1312.0038, 1007.0756, 1405.5137, upcoming]

Gravitational interpretation of bounds

Today:

An example bound

Explore space of 2d CFTs

$$Z\left(-\frac{1}{\tau}\right) = Z(\tau)$$

Impose modular invariance condition

$$\tau \equiv i \exp(s)$$

Expanding condition around fixed point

Evaluate derivatives a

$$\left(\beta\frac{\partial}{\partial\beta}\right)^N Z(\beta)\Big|_{\beta=2\pi} = 0, \quad N \text{ odd}$$

Now write down a partition function

Bounding N - Method

Example: bounding number of states

Consider 2d CFT w/discrete spectrum described by unitary QM

For imaginary τ, CFT partition fcn reduces to thermodynamic partition fcn

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_n \exp(-\beta E_n)$$

Same differential constraints apply; applying derivatives gives constraints

$$\sum_n \exp(-2\pi E_n) g_p(E_n) = 0, \quad p \text{ odd}$$

$$g_p(E_n) \equiv \exp(2\pi E_n)\, (\beta\partial_\beta)^p \exp(-\beta E_n)\Big|_{\beta=2\pi}$$

Explicitly,

$$g_1(E) = -2\pi E$$

$$g_3(E) = -(2\pi E)^3 + 3(2\pi E)^2 - (2\pi E)$$

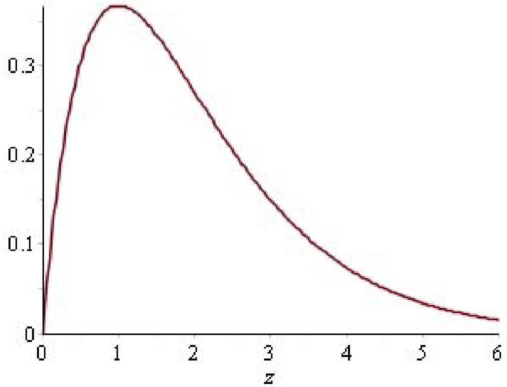
Bounding N - Results

Focus on p = 1 $$\sum_n E_n \exp(-2\pi E_n) = 0$$

Define some energies $E_p \geq 0 \text{ and } E_{p-1} < 0$ and rearrange

$$\sum_{j \geq p} E_j \exp(-2\pi E_j) = \sum_{i=0}^{p-1} |E_i| \exp(2\pi |E_i|).$$

RHS: $$\sum_{i=0}^{p-1} |E_i| \exp(2\pi |E_i|) \leq \sum_{i=0}^{p-1} |E_0| \exp(2\pi |E_0|) = p \frac{c_{\text{tot}}}{24} \exp\left(\frac{\pi c_{\text{tot}}}{12}\right)$$



Sum argument as a function of z = 2 π E

LHS: for large $\mathcal{E}$, count states between $\mathcal{E} e^{-2\pi\mathcal{E}}$ and $\mathcal{E}$

$$N^+_{\mathcal{E}} \mathcal{E} \exp(-2\pi\mathcal{E})$$

Finally: $$N_{\mathcal{E}} \lesssim n \left(\frac{c_{\text{tot}}/24}{\mathcal{E}}\right) \exp\left(\frac{\pi c_{\text{tot}}}{12} + 2\pi\mathcal{E}\right)$$

How good are all of these bounds?

Testing our bounds -- need to generate CFTs with $c > 1$ and ability to control Δ, N: toroidal compactifications

(1) Can we come close to saturating bounds on these quantities?

By studying factorizable CFTs, was shown that lowest primary operator is chiral, saturates bound [Witten '07]

$$1 + \frac{c_{\text{tot}}}{24}$$

A tighter bound...under *quite* an assumption

(2) Can we find examples that obey modular bootstrapping bounds while violating this bound?

Toroidal Compactification - Introduction

Method: from toroidal compactification in string WS theory

Consider $n + n$ (left, right) free scalar fields; $c_{\text{tot}} = 2n$

Compactify theory on some lattice $\Gamma_{n,n}$, investigate spectrum

(placing on this lattice means identifying fields in various directions)

Momenta $p_{L,R}$, of string live on lattice w/ Lorentz. signature; require even, self-dual

Theories have moduli space $\dfrac{O(d,d)}{O(d)\times O(d)}\Big/ O(d,d;\mathbb{Z})$ and can be parameterized using background fields G, B [Narain, Sarmadi, Witten '86]

$$S_d = \frac{1}{4\pi\alpha'}\int d^2z\, \partial_\alpha X^i \partial_\beta X^j \left(\eta^{\alpha\beta}G_{ij} + \epsilon^{\alpha\beta}B_{ij}\right)$$

Toroidal Compactification - Method

What are conformal dimensions? How do we count number of states?

Primary operators will be (derivatives of) scalars and vertex operators

$$V_\phi(z,\bar{z}) = :\prod_i \partial^{m_i} X^{\mu_i}(z) \prod_j \bar{\partial}^{n_j} X^{\nu_j}(\bar{z}) e^{ik\cdot X(z,\bar{z})}:$$

Orbifold to remove low-dimension scalars from spectrum ... ;

Vertex operators have conformal dimensions $\Delta = k^2/2$

Partition fcn: $$Z[G,B] = \frac{1}{|\eta(\tau)|^{2d}} \mathrm{Tr}\left(q^{\frac{1}{2}p_R^2} \bar{q}^{\frac{1}{2}p_L^2}\right)$$

Then maximize length of a given lattice's smallest k^2; instead of $p_{L,R}$, use W^I, K^J

Toroidal Compactification - Method

Relation between $p_{L,R}$, and W^I, K^J:

$$p_{\rm L}^I = \quad W^I + G^{IJ}\left(\tfrac{1}{2}K_J - B_{JK}W^K\right)$$

$$p_{\rm R}^I = \quad -W^I + G^{IJ}\left(\tfrac{1}{2}K_J - B_{JK}W^K\right)$$

In terms of these variables, k^2 found from inner product

$$(W\ K)\,\mathcal{G}^{-1}\begin{pmatrix} W \\ K \end{pmatrix}, \quad \mathcal{G}^{-1} = \begin{pmatrix} 2(G - BG^{-1}B) & BG^{-1} \\ -G^{-1}B & \tfrac{1}{2}G^{-1} \end{pmatrix}$$

Then finding primary conf. dimensions means finding different lengths squared

Then finding number of states corresponds counting lattice-points

==========BREAK============

Toroidal Compactification - Results

What background fields give desired lattice?

(Ex) $n = 1$ ($c_{tot} = 2$):

For $B = 0$ (can't build antisymmetric 1x1); metric goes as

$$\mathcal{G} = \begin{pmatrix} 1/(2R^2) & 0 \\ 0 & 2R^2 \end{pmatrix},$$

At self-dual radius, we maximize minimal vector length

Compare to bounds: $h = \frac{1}{4}, \bar{h} = \frac{1}{4}, \quad \Delta_1 = \frac{1}{2}$

$$0.5 \leq \frac{1}{6} + .4736... = .6403$$

$$0.5 \leq 1.0416$$

Toroidal Compactification - Method

For generic G, we have U(1) x U(1) affine worldsheet algebra

At fixed point of T-duality transformation, enhanced symmetry is SU(2) x SU(2)

Encouraged to investigate maximally enhanced symmetry points….
generalized fixed pts of T-duality group

$$O(d, d; \mathbb{Z})$$

When $B = 0$, enhanced symmetry is $SU(2)^d \times SU(2)^d$

More generally, semi-simple products of ADE type Lie algebras

Maximal symmetry G x G achieved by choosing G ~ Cartan matrix, B appropriately
and is orbifold point in moduli space

(Ex) n = 2

$$G = \frac{\sqrt{3}}{4}\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$ Implies $k^2 = \frac{2}{\sqrt{3}}$ so that $\Delta_1 = 1/\sqrt{3}.$ (.577 vs .807)

Check improvement from turning on *B*:

$$G = -\frac{1}{4}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad B = \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \quad \ell^2_{\min} = \frac{4}{3}, \quad \Delta_1 = \frac{2}{3}$$

$$0.666 \leq 0.8070$$

$$0.666 \leq 1.08333$$

Improvement! ... but considering SU(N) Cartan matrices gives

$$k^2 = \frac{2N}{N+1}.$$

Toroidal Compactification - Results

But: exotic lattices?

In 8 dimensions, try the E8 lattice/Cartan matrix

B = 0 - Min length squared = 2, giving: $\Delta_1 = 1$
Otherwise - Unlike before, fails to improve: $\Delta_1 = 1$

In 24 dims, Leech lattice to the rescue?

B = 0 - Min length squared gives $\Delta_1 = 2$ (holomorphic)
Otherwise ….. ? Seems to give $\Delta_1 = 4$…

But there are issues (factorable?)

(Conjectured in 24k dims: unique self-dual lattice with L_{min} squared = 2k + 2, so that $\Delta_1 = k + 1 = c/24 + 1$ ... improvements?)

Toroidal Compactification - Method

Also interested in counting N(Δ); at self-dual radii, $\Delta_{\mathbf{n},\mathbf{m}} = \frac{1}{2}\sum_{I=1}^{c}(n_I^2 + m_I^2)$

Corresponds to counting integral lattice points inside sphere of radius $\sqrt{2\Delta}$ in $d = 2c$ dimensions (.5235)

$$N(\Delta) \approx \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} r^d = \frac{\pi^c}{\Gamma(c+1)}(2\Delta)^c \approx \frac{1}{\sqrt{2\pi c}}\left(\frac{2\pi e \Delta}{c}\right)^c$$

$$\log N_{c_{\text{tot}}/24} \approx \frac{c_{\text{tot}}}{2}\log\left(\frac{\pi e}{3}\right) \approx 0.523059\ c_{\text{tot}}.$$

For Δ ~ c, this approximation breaks (surface area grows more rapidly with dimension than interior)

Well-studied problem, use/generalize results of [Mazo, Odlyzko, '90]

Toroidal Compactification - Results

Proper counting gives (.5235)

$$\lim_{n\to\infty} \log(\max N)/n = 0.566251$$

$$n_{\frac{c_{tot}}{24}} \approx e^{0.344606 c_{tot}}$$

Hypercubic success...what about E8, Leech? Also success

Consider larger extremal self-dual lattices?

BTZ black hole entropy ~ exp(π ctot / 6)

(π ctot / 6 ~ 0.523598)

Seems like these lattices cannot be boundary theories for 3D gravity theories (not enough entropy)

k	$c_L(c_R) = 24k$	$\lim \log(\max N)/n$
1	24	0.529435
2	48	0.525423
3	72	0.523599
4	96	0.523546
5	120	0.523314

Conclusion

(1) Derived bounds on conformal dimensions, numbers of states/primaries

(2) Generalized work of others to consider theories w/out sparse light spectra

(*) Candidate CFT showing tighter bounds on dimensions are unlikely

(*) Ruling out extremal self-dual CFTs as boundary theories of 3D gravity

Thanks!

END

(extra slides)

Bounding Δ_n - Summary

Found upper bounds on conformal dims of lightest few states; thus found upper bounds on masses of lightest states in dual gravitational theory (when it exists)

With appropriate constraints, can bound *n* operators; so there exist at least *n* states obeying conformal dim/mass upper bounds

Found lower bound on number of states; upper bound?

Independently explored by others [Hellerman and Schmidt-Colinet '11, Hartman, Keller, Stoica, '14]

We provide alternate arguments--more general in some ways, weaker in others

Bounding N - Results

Calculate some interesting limits of $$N^{+}_{\mathcal{E}}\mathcal{E}\exp(-2\pi\mathcal{E}) < p\frac{c_{\text{tot}}}{24}\exp\left(\frac{\pi c_{\text{tot}}}{12}\right)$$

$$N^{*}_{\mathcal{E}} < n\left[1+\left(\frac{c_{\text{tot}}/24}{\mathcal{E}}\right)\exp\left(\frac{\pi c_{\text{tot}}}{12}+2\pi\mathcal{E}\right)\right]$$

$$N_{\mathcal{E}} \lesssim n\left(\frac{c_{\text{tot}}/24}{\mathcal{E}}\right)\exp\left(\frac{\pi c_{\text{tot}}}{12}+2\pi\mathcal{E}\right)$$

Comparison with other work [Hartman, Keller, and Stoica '14, Hellerman and Schmidt-Colinet '11]]

$$\log N_{\mathcal{E}} < \log n + \log\left(\frac{c_{\text{tot}}/24}{\mathcal{E}}\right) + \frac{\pi c_{\text{tot}}}{12} + 2\pi\mathcal{E}.$$

$$S(E) \lesssim \frac{\pi c_{\text{tot}}}{12} + 2\pi E\left(\epsilon < E < \frac{c_{\text{tot}}}{24}\right)$$

and $$N_{marg}(c_{\text{tot}}) \leq \frac{n c_{\text{tot}}}{48-c_{\text{tot}}}e^{4\pi}\left(1+e^{2\pi}\frac{\delta(c_{\text{tot}})}{1+\delta(c_{\text{tot}})}\right)$$ vs. $$N < \left(\frac{c_L+c_R}{48-c_L-c_R}\right)\cdot\exp\{+4\pi\} - 2$$

Gravity - Dictionary

AdS/CFT: equivalence of string theory on AdS background (Λ < 0), CFT on boundary [Maldacena '98] [MAGOO, '00] [Witten, '98]

Study of asymptotically AdS$_3$ spacetimes lead to [Brown, Henneaux '86]

$$c + \bar{c} = \frac{3L}{G_N} \qquad (\; L = |\Lambda|^{-1/2} \;)$$

Match spectrum of bulk objects w/ boundary primaries $E^{(rest)} = \frac{\Delta}{L}$

Bounds now become $M_n \leq M_n^+ \equiv \frac{1}{L}\Delta_n^+|_{c_{\text{tot}}=\frac{3L}{G_N}}$

Evaluate: $M_n \leq \frac{1}{4G_N} + \frac{D_n}{L}$; in flat space limit, $M_n \leq \frac{1}{4G_N}$

Bound on number of states implies bound

$$\frac{\pi L}{4G_N} + O\left(\log \frac{L}{G_N}\right) \leq \log N$$

Upper bound on states gives upper bound on primaries

For "pure" gravity, $$\log(N^{+}_{c_{\text{tot}}/24}) \leq \frac{\pi c_{\text{tot}}}{6} + O(\sqrt{c_{\text{tot}}})$$

becomes $$\log N \leq \frac{\pi L}{2G_N} + O\left(\sqrt{\frac{L}{G_N}}\right)$$

Thus:

$$\frac{\pi L}{4G_N} + O\left(\log \frac{L}{G_N}\right) \leq \log N \leq \frac{\pi L}{2G_N} + O\left(\sqrt{\frac{L}{G_N}}\right)$$