Interaction-induced Symmetry Protected Topological Phase in Harper-Hofstadter models

Lode Pollet
Dario Hügel
Hugo Strand, Philipp Werner (Uni Fribourg)
Algorithmic developments

diagrammatic Monte Carlo, (bosonic) cluster methods, …

the quest for novel numerical methods going beyond the state of the art:
Harper-Hofstadter model

Hardcore bosons

\[ H_\Phi = -\sum_{x, y} \left( t_x e^{i\frac{\pi}{2}} b^\dagger_{x+1, y} b_{x, y} + t_y b^\dagger_{x, y+1} b_{x, y} \right) + h.c. \]
Hall effect

\[ R_H = \frac{E_y}{j_x B_z} = -\frac{1}{ne} \]

Integer Quantum Hall effect

Landau levels
topological invariant: first Chern number (TKNN)

\[ \sigma_{xy} = ne^2 / h \]

\[ n = \frac{1}{2\pi i} \int_{\text{BZ}} d^2 k \langle \nabla_k u(k) \rangle \times |\nabla_k u(k)\rangle \]
spin-Quantum Hall Effect

$Z_2$ topological invariant protected by time reversal symmetry
(TKNN integer is 0)

eg: spin-orbit coupling in graphene (but too weak),
CdTe/HgTe/CdTe structures (band inversion as function of thickness)

(CL Kane)
cold atom experiments

- very strong effective magnetic fields
- all optical setup with bosonic atoms
- Chern number has been measured (also for hexagonal lattice with fermions (ETHZ))
- add interactions?

how to study?

The usual path integral Monte Carlo simulations do not work because of the infamous sign problem…

intermezzo: develop an approximate method and benchmark it
classical Ising (ferromagnet $J > 0$):

we are interested in the magnetization on every site:

Weiss field:

approximation:

selfconsistency equation:

mean-field theory

$$H = -J \sum_{\langle i,j \rangle} S_i S_j + h \sum_i S_i$$

$$m_i = \langle S_i \rangle$$

$$H_{\text{eff}} = - \sum_i h_i^{\text{eff}} S_i$$

$$\beta h_i^{\text{eff}} = \tanh^{-1} m_i$$

$$h_i^{\text{eff}} \approx h + \sum_j Jm_j = h + zJm$$

$$m = \tanh(\beta h + z\beta Jm)$$
We want to develop the dynamical mean-field solution for the 3d Bose-Hubbard model.

\[ H = -t \sum_{\langle i,j \rangle} b_i^\dagger b_j + \frac{U}{2} \sum_i n_i(n_i - 1) - \mu \sum_i n_i \]

write down single-site action:

\[ S_{\text{imp}} = \int_0^\beta d\tau b^\dagger(\tau) [\partial_\tau - \mu] b(\tau) + \frac{U}{2} \int_0^\beta d\tau n(\tau) [n(\tau) - 1] \]

we lose the momentum dependence for the self-energy.

works fine for normal phase and Mott phase.

we want:

include the physics of the weakly interacting Bose gas non-perturbatively, which is the limit \( t \gg U \)

we want: correction on top of mean-field
let's add a symmetry breaking field:

\[-z t \phi \int_0^\beta d\tau [b(\tau) + b^\dagger(\tau)]\]

this is the same as in static mean-field which can produce a condensate

**Bogoliubov** prescription:

\[b(\tau) = \langle b\rangle + \delta b(\tau)\]

imag time dynamics can be added in the two-particle channel.

The second source field can only couple to the normal bosons, otherwise

*double counting* will occur (Nambu notation):

\[-\frac{1}{2} \int_0^\beta d\tau d\tau' \delta b^\dagger(\tau) \Delta(\tau - \tau') \delta b(\tau')\]

which contains normal and anomalous propagators.

see J.W. Negele and H. Orland, Quantum Many-Particle Systems (Addison-Wesley Publishing Company 1988) ISBN 0-201-12593-5 for how to treat broken symmetry

Final step: re-express \(\delta b\) in terms of full \(b\)
weakly-interacting Bose gas

*Why BDMFT should be good:* look at self-energies of weakly interacting Bose gas (Beliaev)

\[
\Sigma(P) = -2G(r = 0, \tau = -0)U + 2n_0U = 2nU
\]

\[
\tilde{\Sigma}(P) = n_0U.
\]

momentum independent to leading order


\[
G(P) = -\frac{\nu + \mu_0}{\xi^2 + E^2(k)}
\]

\[
F(P) = \frac{\mu_0}{\xi^2 + E^2(k)}
\]

\[
E^2(k) = \epsilon(k)[\epsilon(k) + 2\mu_0]
\]

\[
\tilde{\mu} = \mu - 2nU
\]

similar in magnitude at low temperature, but opposite in sign
comparison in 3 dimensions

![Graph showing the comparison of B-DMFT, Monte Carlo, and Mean Field methods for kinetic energy $E_{\text{kin}}/t$ as a function of $T/t$. The graph includes a zoomed-in inset focusing on the region around $T/t = 5$.](image-url)
phase diagram in 3 dimensions

finite temperature, unit density

ground state

finite temperature, unit density

ground state

phase diagram in 3 dimensions

finite temperature, unit density

ground state

finite temperature, unit density

ground state
results in two dimensions

finite temperature, unit density

ground state

normal

SF

Mott

SF

Mott

Mott

Bethe z=4

Mean Field
Bosonic self-energy functional theory

only 3 parameters needed, determined variationally accuracy better than 1% for Bose-Hubbard model in every parameter regime

D. Hügel and L. Pollet, PRB 2015
Harper-Hofstadter model

\[ H_\Phi = - \sum_{x,y} \left( t_x e^{i y \Phi} b_{x+1,y}^{\dagger} b_{x,y} + t_y b_{x,y+1}^{\dagger} b_{x,y} \right) + h.c. \]
Chern numbers for non-interacting problem

diagonalize single-particle $H$:

$$H_\Phi = \int \frac{dk dq}{2\pi} \left( \vec{v}_{k,q} \cdot \vec{h}_{k,q} \right)$$

$$\vec{v}_{k,q} = \begin{pmatrix} -2t_x \cos(k) \\ -2t_x \cos(k - \frac{\pi}{2}) \\ -2t_y \cos(q) \end{pmatrix}$$

$$\vec{h}_{k,q} = \begin{pmatrix} A_0(k,q) \\ A_1(k,q) \\ B(k,q) \end{pmatrix}$$

$$A_l(k,q) = n_l(k,q) - n_{l+N_x/2}(k,q)$$

$$B(k,q) = \frac{e^{-iq}}{2\cos(q)} \sum_l b_{l+1}^\dagger(k,q)b_l(k,q) + h.c.$$
real-space cluster mean-field
studied several fluxes
did not look at hopping anisotropy
found metastable (f)QH phases, almost degenerate with the superfluid
many density-wave instabilities
our method

• cluster mean-field but in momentum space

• hence: simpler than BDMFT or SFT (only $\Phi$, no pair terms)

• impurity problem (4x4) solved with Lanczos

• no ‘connected’ Green function of non-condensed particles of original model
do NOT break translational invariance by working with clusters in momentum space instead of real space (but simpler than selfenergy functional theory):

\[(K, Q), (\tilde{k}, \tilde{q})\]
\[(X, Y), (\tilde{x}, \tilde{y})\]
coarse grain the dispersion:

\[
\bar{\epsilon}_{K,Q} = \frac{N_c M_c}{N M} \sum_{\tilde{k}, \tilde{q}} \epsilon_{K + \tilde{k}, Q + \tilde{q}}
\]

divide the Brillouin zone into patches: this breaks up Hamiltonian:

\[H = H_c^{\text{intra}} + \Delta H_c^{\text{inter}}\]

care is needed in case of symmetry breaking:

\[\phi_{K,Q}(\tilde{x}, \tilde{y}) = \phi_{K,Q}\]

cluster-Hamiltonian can be written as

\[
H'_{\text{eff}} = \sum_{X', Y', X, Y} \bar{t}_{(x', y'), (x, y)} b_{X', Y'}^\dagger b_{X, Y} + \mu \sum_{X, Y} n_{X, Y} + \frac{U}{2} \sum_{X, Y} n_{X, Y} (n_{X, Y} - 1) + \sum_{X, Y} (b_{X, Y}^\dagger F_{X, Y} + F_{X, Y}^* b_{X, Y})
\]

\[F_{X, Y} = \sum_{x, y'} \delta t_{(x, y'), (x, y)} \phi_{x', y'}^{(x, y)}\]

\[\phi_{x, y} = \langle b_{x, y} \rangle\]

\[
\bar{t}_{(x', y'), (x, y)} = \frac{1}{N_c M_c} \sum_{K, Q} e^{i(K(x' - x) + Q(y' - y))} \bar{\epsilon}_{K, Q}
\]

\[
\delta t_{(x', y'), (x, y)} = t_{(x', y'), (x, y)} - \bar{t}_{(x', y'), (x, y)}
\]
benchmarking

2d Bose Hubbard model, no anisotropy, no flux (MF: mean-field; CG cluster Gutzwiller)

2d Bose Hubbard model, no flux; black = half filling

chiral ladder system
ground state phase diagram

non-degenerate gapped SPT phase
$U(1) \times Z_2^T$
has $Z_2$ index;
reminiscent of quantum spin Hall
symmetry operation: $\mathcal{T}U_\theta \mathcal{T}U_\theta = e$

FQHE: found, but all unstable against superfluidity

https://arxiv.org/abs/1205.3156
compare with free fermions:

SPT at filling 1 strongly reduced
SPT at filling 2 absent for free fermions
Chern numbers for interacting problem

twisted boundary conditions:

\[
C = \frac{1}{2\pi} \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \left( \partial_{\theta_x} A_y - \partial_{\theta_y} A_x \right)
\]

\[
A_j(\theta_x, \theta_y) = i \langle \Psi(\theta_x, \theta_y) | \partial_{\theta_j} | \Psi(\theta_x, \theta_y) \rangle
\]

\[
T_{x/y} \Psi(\theta_x, \theta_y) = e^{i\theta_{x/y}} \Psi(\theta_x, \theta_y)
\]

\[
t_y \rightarrow t_y e^{i\theta_y/L_y}
\]

in the thermodynamic limit reciprocal space is continuous, and the phase twist infinitesimal

\[
\vec{v}_{k,q} \rightarrow \vec{v}_{k,q}(\theta_x, \theta_y) = \begin{pmatrix}
-2t_x \cos \left( k - \theta_x/L_x \right) \\
-2t_x \cos \left( k - \theta_x/L_x - \frac{\pi}{2} \right) \\
-2t_y \cos \left( q - \theta_y/L_y \right)
\end{pmatrix}
\]

this is just a momentum shift for every momentum

\[
\langle \Psi(\theta_x, \theta_y) | \hat{h}_{k,q} | \Psi(\theta_x, \theta_y) \rangle = \langle \Psi(0,0) | \hat{h}_{k+\theta_x/L_x,q+\theta_y/L_y} | \Psi(0,0) \rangle
\]

we hence look at the winding of $h$ projected on the space of hard-core bosons
coincidence?

project interacting problem onto non-interacting bands:

\[ n = 1/4 \ (\nu = 1) \]

occupation numbers:

\[ \nu_0 = 1, \nu_1 = 0, \nu_2 = 0 \]

observe:

\[ c_0 \nu_0 = -1 \]

\[ n = 1/2 \ (\nu = 2) \]

occupation numbers:

\[ \nu_0 = 1.45, \nu_1 = 0.25, \nu_2 = 0.05 \]

observe:

\[ \nu_0 c_0 + \nu_1 c_1 + \nu_2 c_2 \approx -1.45 + 0.5 - 0.05 = -1 \]

(the result of this procedure is 0 for the trivial band insulators)

(not the same as the approach by T. Neupert et al)
Summary

- Bosonic dynamical mean-field theory, bosonic self-energy functional theory
- Cluster extensions
- SPT phases in interacting Harper-Hofstadter models; one purely due to interactions and of quantum spin-Hall like nature
- Perhaps the easiest around to check experimentally
- Checks: extend to selfenergy functional methods, other fluxes, seeing topological phase transition?
- Many interesting extensions possible (disorder, dynamics)